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Equilibrium distributions for random walkers in random media

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Abstract. We study the Renyi entropies of the probability distribution of a random walker in a random medium of finite total volume. In one dimension of space we show that they tend, for large lattices, to constants independent of the lattice size. Thus, the distribution is not multifractal, in contrast to a recent claim. We conjecture that the same is true for any dimension if the random force is derived from a potential. In the more interesting general case (unconstrained forces), we cannot make a definite statement.

1. Introduction

Random walks in random media have been the subject of numerous recent papers [1-10]. The central quantity studied in most of them was the RMS displacement $(\Delta x)^2$ and its time dependence in an infinitely extended medium. In the following, we shall only consider the case of zero average drift and of short-range correlations between the forces. In one dimension one finds [10] a logarithmic increase $(\Delta x(t))^2 \sim (\log t)^4$. In two or more dimensions, the results depend on whether the random forces are curl-free (and thus derivable from a potential), are divergence-free, or are completely general. In the latter case, the randomness does not change the usual asymptotic behaviour $\Delta x(t) \sim \sqrt{t}$ in three and more dimensions, while it adds logarithmic corrections to it in two dimensions [3, 7]. In the potential case, the randomness is still relevant in two dimensions, and the critical dimension above which normal diffusion prevails is greater than 2 [4, 6].

In contrast to the time dependence of the displacement, much less effort has been devoted to understanding the ultimate distribution in media of finite volume.

In one dimension, this might seem a trivial problem: any random medium can be represented by a random potential, and the distribution is simply given by the Boltzmann-Gibbs formula. Nevertheless, it is not entirely trivial to characterise the statistical properties of this distribution. The only paper we are aware of which deals with this question [9] arrives at a wrong result. It claims that the distribution is multifractal [11]. This would mean that the Renyi entropies H_q [12] increase linearly with the logarithm of the length L of the medium, such that the generalised dimensions [13] $D_q = H_q/\log L$ are finite and depend non-trivially on q.

Such a multifractal behaviour seems at first sight very reasonable. Indeed, the height distribution of a random potential is Gaussian, and thus the distribution of the probabilities $p(x)$ for the walker to be at any randomly chosen position x should be log-normal. A log-normal distribution is the simplest multifractal distribution (we ignore here the usual problems associated with the lack of normalisation of a log-normal

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distribution; actually the distribution should be log-binomial according to this argument).

It is the purpose of the present paper to show that this is wrong. The point missed by the above argument is that the distribution is not self-averaging. It would be correct if we were to study a walker in a potential with a Gaussian distribution *but with short-range spatial correlations.* But assuming a random short-range force or a random bias in a hopping model forces the potential to be the graph of a Brownian walk, thus implying very strong correlations of the potential.

As a consequence, one finds that all the Renyi entropies with $q>0$ tend to constants for $L\to\infty$, in the one-dimensional case.

We conjecture that this behaviour prevails also in higher dimensions, provided the random forces are derived from a potential, although our arguments are much weaker there than in one dimension. Furthermore, in general the forces cannot be derived from a potential (if they are not curl-free). In this case, we have no good theoretical arguments. Also, numerical simulations are much harder than in one dimension and thus our numerical results for this case are inconclusive.

2. Theoretical arguments

Let us consider a finite array of N sites labelled by $i=1,2,\ldots,N$. At each time step, a particle at site i will jump either to its right neighbour (with probability u_i) or to its left neighbour (with probability $1-u_i$). We assume that for $i\neq 1, N$ these hopping rates are random variables with distributions independent of i, and with finite logarithmic variance, $\langle(\ln u)^2\rangle<\infty$. In addition, we assume left-right symmetry, i.e. we assume u_i and $1-u_i$ to have the same distribution (in particular, $\langle u\rangle=\frac{1}{2}$).

For a fixed configuration of hopping rates, the probability $p_i(t)$ to be at site i thus evolves according to the master equation

$$p_i(t+1)=u_{i-1}p_{i-1}(t)+(1-u_{i+1})p_{i+1}(t). \tag{2.1}$$

If we assume no-flux boundary conditions, $u_1=1$ and $u_N=0$, then we have detailed balance for the stationary distribution

$$u_ip_i=(1-u_{i+1})p_{i+1}. \tag{2.2}$$

We then obtain immediately

$$p_1=Z^{-1}$$

$$p_i=Z^{-1}\prod_{k=1}^{i-1}\frac{u_k}{1-u_{k+1}} \qquad \text{for } i\geqslant 2 \tag{2.3}$$

where the normalisation constant Z is given by

$$Z=1+\sum_{i=2}^{N}\prod_{k=1}^{i-1}\frac{u_k}{1-u_{k+1}}. \tag{2.4}$$

For large N, we can write $\log(Zp_i)$ approximately as a sum of independent terms

$$\ln(Zp_i)=\sum_{k=1}^{i}\ln\frac{u_k}{1-u_k}. \tag{2.5}$$

Its increase with i is just like a random walk with diffusion constant

$$D=\frac{1}{2}\left\langle\left(\ln\frac{u_k}{1-u_k}\right)^2\right\rangle. \tag{2.6}$$

The lack of self-averaging mentioned in the introduction arises from the fact that the normalisation Z is in general not of order N. Instead, its logarithm diverges in the average like $\sqrt{ND}$.

Our aim is to estimate the average Renyi entropies defined as

$$H_q = \left\langle (1-q)^{-1} \ln \sum_{i=1}^{N} p_i^q \right\rangle \tag{2.7}$$

with $q > 0$, in the limit of large N.

The dominant behaviour of H_q can indeed be estimated very easily. We first observe that the maximum of p_i in a given realisation of the lattice is almost surely (a.s.) far from the ends, for large N, and the probabilities p_i will be non-negligible only near that maximum. Around the maximum, the logarithm of p_i looks thus typically as shown in figure 1. Since $\log p_i$ essentially makes a random walk with diffusion constant D, the typical range over which p_i is non-negligible is of length $\sim D$. Thus we immediately get a rough estimate $H_q \simeq -\ln D$ for all $q > 0$.

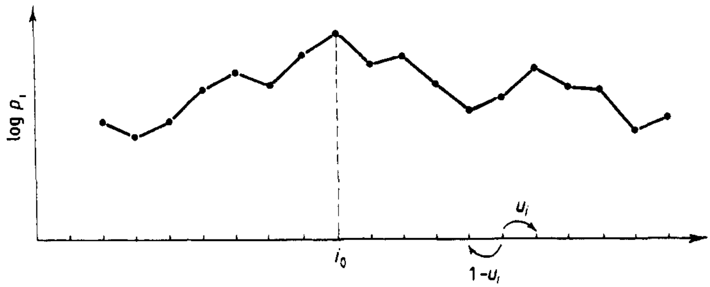

Figure 1. Typical behaviour of the logarithm of the stationary probabilities p_i. The highest probability is assumed to occur at site i_0.

Let us now replace this rough argument by a more detailed one which involves essentially a mean-field-type approximation. We consider only the limit of large N and of hopping rates close to $\frac{1}{2}$. The latter corresponds to small differences $p_i - p_{i-1}$, and thus allows us to approximate the random function $i \to p_i$ by a continuous random walk.

For a given configuration of hopping rates, we denote the position of the maximal p_i by i_0 (see figure 1), and we define

$$\phi(r) = \ln(p_{i_0+r}). \tag{2.8}$$

Just as $\log p_i$ is, in the considered limit, a random walk in a homogeneous medium with diffusion constant D starting at $i = 0$, $\phi(r)$ is in this limit a random walk starting at $r = 0$, *but conditioned on staying below the value* $\phi(0)$ *for all 'times'* r. The average distance for this kind of random walk is found by the following argument. First we have to estimate the $\Delta\phi$ dependence of the probability distribution $P(\Delta\phi, r)$, $\Delta\phi = \phi(0) - \phi(r)$. This distribution is, up to normalisation, the product of the conditional probability ($\propto \Delta\phi \exp(-\Delta\phi^2/4Dr)$) conditioned on having $\Delta\phi(r') \geq 0$ for $0 < r' < r$, times the probability ($\propto \Delta\phi$ [14]) that $\Delta\phi(r'') > 0$ for $r < r'' < R$, for some $R \gg r$. From this, the average $\Delta\phi$ is found to increase with r as

$$\langle \phi(0) - \phi(r) \rangle = \left(\frac{16}{\pi} Dr \right)^{1/2}. \tag{2.9}$$

The mean-field-type approximation consists now of neglecting all fluctuations of $\phi(r)$, i.e. in assuming that the brackets in (2.9) can be omitted. In this approximation we get

$$\lim_{N\to\infty} H_q \simeq (1-q)^{-1}\ln\left(2\int_0^\infty \mathrm{d}r\, p_{i_0+r}^q\right) \tag{2.10}$$

$$= (1-q)^{-1}\ln\left(2\int_0^\infty \mathrm{d}r\, Z^{-q}\exp(-q\sqrt{16Dr/\pi})\right) \tag{2.11}$$

with

$$Z = 2\int_0^\infty \mathrm{d}r\,\exp(-\sqrt{16Dr/\pi}). \tag{2.12}$$

Evaluating this gives straightforwardly

$$\lim_{N\to\infty} H_q \simeq \frac{2\ln q}{q-1} - \ln\frac{4D}{\pi}. \tag{2.13}$$

We recover indeed the leading term $H_q \simeq -\ln D$ as long as $q>0$. We also see that $H_q \to \infty$ for $q\to 0$, as it should be (we have of course $H_0 = \ln N$).

Neglecting fluctuations should not induce a very large error for $q \simeq 1$, but is obviously wrong for large q. The limit $q\to\infty$ can, however, be treated exactly, as we get in this limit only contributions from the fastest possible decay of $\phi(r)$. Assume that the u_i are bounded away from zero, so that $\alpha \equiv \ln\sup[(1-u)/u]$ is finite. Then the dominant contribution to H_∞ comes from a configuration where $\phi(0)=\phi(1)$, $\phi(1)-\phi(r)=\alpha(r-1)$ for $r>1$, and $\phi(0)-\phi(r)=-\alpha r$ for $r<0$. An easy calculation then gives

$$\lim_{q\to\infty}\lim_{N\to\infty} H_q = \ln\frac{2}{1-\mathrm{e}^{-\alpha}}. \tag{2.14}$$

3. Numerical simulations and discussion

In order to test the predictions of the last section, we have performed extensive numerical simulations.

These simulations are again easiest for one-dimensional lattices, since they enable us to use (2.3) and (2.4). These allow very fast evaluation on very large lattices. The main restriction on the lattice size is that numerical overflow occurs on most compilers for lattices of $\sim 10^4$–10^5 sites.

In figure 2, we show results obtained from lattices of up to 15 000 lattice sites, with 15 000 realisations for each size (indeed, the smaller lattices were sublattices of the largest ones, which helped to keep CPU time low). The hopping rates were chosen uniformly from the interval $0.3 < u_i < 0.7$. This gives a diffusion constant $D = 0.1142$, and a limiting Renyi entropy $\lim_{N\to\infty} H_\infty = 1.253$. Each curve in figure 2 corresponds to a fixed value of q, with $0 \leqslant q \leqslant 3$. Since we anticipate the leading finite-size correction to the Renyi entropies to go like constant $\sqrt{N}$, we plotted them against $1/\sqrt{N}$. We

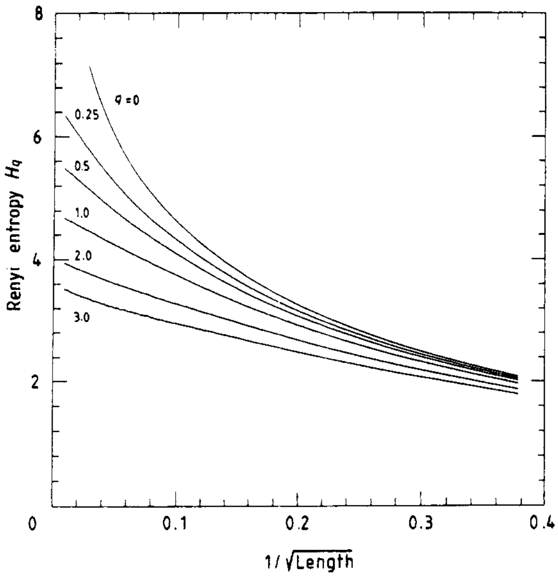

Figure 2. Renyi entropies (in natural units) for walkers on lattices with up to 1.5×10^4 sites, and with a uniform distribution of hopping probabilities in the range $0.3 < u < 0.7$. For each size, the data are averaged over 1.5×10^4 different realisations. The curves (starting from the topmost) correspond to $q = 0$, 0.25, 0.5, 1.0, 2.0 and 3.0.

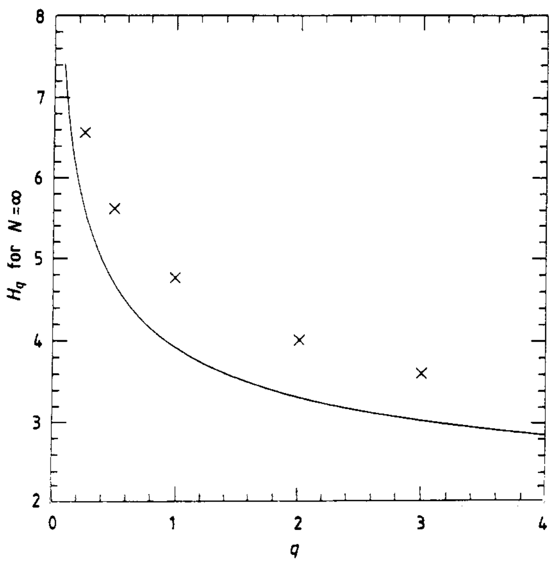

Figure 3. Renyi entropies (in natural units) for walkers on infinite one-dimensional lattices. Crosses are extrapolated results from figure 2, while the full curve corresponds to (2.13).

find indeed that the data can be extrapolated linearly to $N\to\infty$, except for those corresponding to $q=0$. The extrapolated values are shown in figure 3, together with our prediction (2.13). The agreement is reasonably good, in view of the crudeness of the approximations involved in the theoretical predictions. Most of the discrepancy can indeed be attributed to the fact that there are strong finite-r corrections to (2.9) in the case of finite diffusion constant D.

In [9], the conclusion that the distribution is multifractal was not based on simulations using (2.3) and (2.4). Instead, there the time-dependent master equation (2.1) was solved by iteration, with $p_i(t=0)=\delta_{i0}$ as initial condition. Since this is much slower, these authors could only reach distributions which were spread over ~50–100 lattice sites. From our data we see that for lattices of this size, we would not be able to reliably distinguish between a multifractal distribution and the correct one. We might add that the authors of [9] did not study the asymptotic distribution for $t\to\infty$ on a finite lattice (as we did), but rather the distribution for large but finite t on an infinite lattice. Their conclusion would nevertheless have implied that the distributions we have studied were also multifractal.

While the detailed equations (2.13) and (2.14) are, of course, specific to one dimension of space, the general heuristic argument for the finiteness of the Renyi dimensions should also hold in any dimension $\geqslant 2$ if the random force is derived from a potential. The reason for this is that in any dimension the decay of $\log p_i$ away from its maximum should be essentially $\simeq\exp(-\sqrt{r})$, and this decay cannot be overcome by the volume element which increases only like a power of r.

The situation is much less clear in the general case where the force is not derivable from a potential. In this case, there is an upper critical dimension $d_c=2$ above which the randomness does not imply any anomalous diffusion [7]. This would suggest that $d=2$ is also the critical dimension at which the effect of the randomness on the clustering vanishes.

In order to test this, we have performed simulations on two-dimensional square lattices with $L\times L$ sites, with $L\leqslant 128$. The rates for hopping in the ith direction $(i=1,\ldots,4)$ were chosen as $u_i=r_i/\Sigma_{j=1}^4 r_j$, with r_j uniformly $\in[0.1, 0.9]$. Unfortunately, in this case there exists no closed expression for the equilibrium distribution similar to (2.3). Thus we had to estimate it by simulation or by a relaxation method. We found that direct simulations of the walks gave very slow convergence. Also, over-relaxation methods could not be applied, as the eigenvalues governing the relaxation are not bounded away from zero. Thus we finally used the simple Gauss-Seidel method. Since this is rather slow, our statistics are now much worse than in one dimension. Also, we should expect the asymptotic behaviour to set in later. As a consequence, our data (shown in figure 4) are much less clear than those for the one-dimensional case. Nevertheless, they indicate that the distribution might not be multifractal in the present case either. In order to see this more clearly, we have shown in figure 4 not the Renyi entropies H_q themselves, but rather the effective dimension $D_q(L)=H_q/\log L$. These should tend towards a constant for $L\to\infty$ if the distribution were multifractal. Instead, there is a marked downward trend for large L. We checked that this behaviour cannot be fitted by a power-law correction $D_q(L)=D_q+\text{constant}/L^\alpha$, for any value of α. Of course, the data do not show that the Renyi entropies are independent of the size L either. Our conclusion is thus that we see either strong logarithmic corrections to multifractality (as we might expect at a critical dimension), or that the Renyi entropies behave essentially different from what might have been expected on the basis of the time dependence of diffusion.

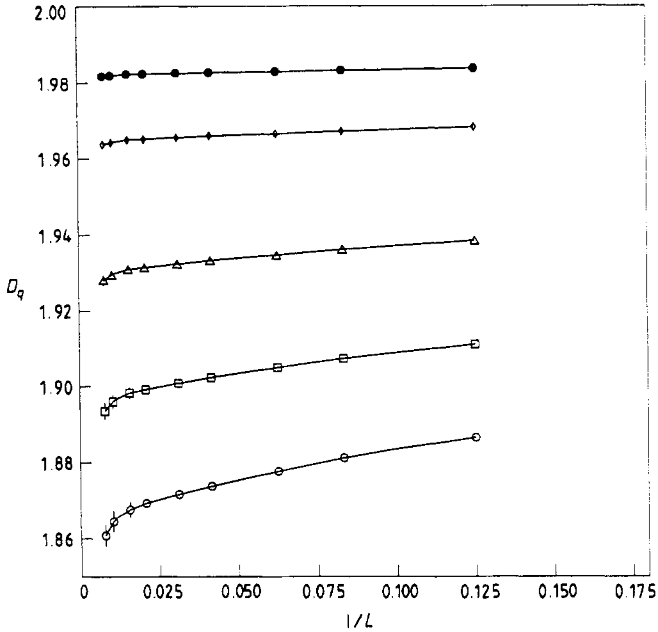

Figure 4. Estimated generalised dimensions $D_q(L) = H_q/\log L$ for walks on 2D square lattices of size $L \times L$ with periodic boundaries, plotted as function of $1/L$. Curves are plotted for $q = 4$ (○), $q = 3$ (□), $q = 2$ (△), $q = 1$ (◇) and $q = 0.5$ (●).

Note added in proof. After submitting this paper, we have performed extensive additional simulations of the 2D model, with $L = 128$, $L = 192$ and $L = 256$. These new data suggest that D_q do tend towards finite values for $L -> \infty$, in contrast to those in figure 4. In particular, D_q $(L = \infty)$ is 1.865 ± 0.005 for $q = 4$, 1.896 ± 0.003 for $q = 3$, 1.930 ± 0.002 for $q = 2$, and 1.965 ± 0.001 for $q = 1$.

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